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PHENOMENOLOGY OF INERTIA EFFECTS IN A DISPERSED SOLID-FLUID MIXTURE

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Abstract—Combining single particle results, average equations and thermodynamic considerations, we propose a way to build the equations describing a suspension of rigid spherical particles in a carrier fluid, with emphasis on inertia effects including virtual mass. The spatial fluctuations of the fluid velocity field are depicted by two phenomenological functions $f(\alpha_s)$ and $g(\alpha_s)$ of the particle volume fraction, and a third function $h(\alpha_s)$ is necessary to describe the intensity of the particles internal stress. It is shown that all inertia effects occurring in the relative translational motion can be derived from the two functions f and g–h only. The conditions under which the above system of equations is hyperbolic are determined and comparison is made with what is presently known about f, g and h in the dilute limit.

1. INTRODUCTION

The forces acting on particles in a dispersed mixture mostly have a viscous origin, but sometimes inertia forces do play a role. This is the case when flows are accelerated, either due to nonstationarity (propagation of sound) or to spatial non-uniformities (flow through a throat). Our knowledge of inertia effects is still fragmentary and we often reduce them to virtual mass effects although other terms are intuitively felt to exist, for instance when one adds *ad hoc* forces in the transfer of momentum between phases. The point is that the complete knowledge we now have of the motion of a single particle in a ideal fluid (van Beek 1982 and Voinov 1973) is not sufficient to derive equations for a mixture, even a dilute one. In this paper we suggest that, to get a reasonable set of equations, it is necessary to combine single particle results, average equations and a general continuum approach of mixtures.

In section 2 the thermodynamic approach of a two-velocity mixture is presented. For the sake of simplicity, we restrict ourselves to the case of spherical rigid particles dispersed in a carrier fluid. In sections 3 and 4, we determine the consequences of the necessary compatibility between the above thermodynamic equations and the average equations commonly used in the multiphase flow community. In section 5, we transform our equations so as to get an expression for the interphase momentum exchange that can be compared with previous works. Section 6 deals with the problem of characteristics while a summary and a discussion of our results are presented in the concluding section.

2. THERMODYNAMICS OF A SOLID-FLUID MIXTURE WITH VIRTUAL MASS EFFECTS

2.1 *The local and average thermodynamic relations for the fluid*

The unit mass of a pure fluid phase is described by a Gibbs equation

$$\mathrm{d}\tilde{\varepsilon}_f = \tilde{T}_f\,\mathrm{d}\tilde{s}_f - \tilde{p}_f\,\mathrm{d}\left(\frac{1}{\tilde{\rho}_f^0}\right), \qquad [1]$$

and a Gibbs-Duhem relation

$$\frac{1}{\tilde{\rho}_f^0}\,\mathrm{d}\tilde{p}_f = \mathrm{d}\tilde{\mu}_f + \tilde{s}_f\,\mathrm{d}\tilde{T}_f, \qquad [2]$$

where $\tilde{\rho}_f^0$, $\tilde{p}_f$, $\tilde{T}_f$, $\tilde{\epsilon}_f$, $\tilde{s}_f$ and $\tilde{\mu}_f$ are respectively the fluid mass per unit volume, pressure, temperature, specific internal energy, specific entropy and chemical potential or free enthalpy per unit mass, i.e.

$$\tilde{\mu}_f = \tilde{\epsilon}_f - \tilde{T}_f\tilde{s}_f + \frac{\tilde{p}_f}{\tilde{\rho}_f^0}. \qquad [3]$$

The thermodynamic relations [1] to [3] are valid at a local or "microscopic" level, that is to say on a scale much smaller than the dimension of a particle. We are in fact interested in average thermodynamic relations, valid in any volume of the mixture containing many particles (the so-called mesoscale volume) but still being a "point" as compared to macroscopic volumes. In general, all the fluid variables vary from point to point around the particles but we want here to show that, for solid particles, the spatial inhomogeneities of $\tilde{\rho}_f^0$ and $\tilde{T}_f$ are negligible and that the average relations then take a rather simple form.

When a particle moves relatively to a fluid, the fluid develops on the particle a force which deforms it and changes its volume. In turn, this volume change (if rapid enough) will modify the mean particle temperature. However, if the particles are solid (or even liquid) the volume change is so tiny that one never expects a noticeable difference with the mean temperature of the surrounding fluid. In other words, provided there is no phase change and that the particles are not bubbles, the inhomogeneities of $\tilde{T}_f$ around the particles are negligible. It must be stressed that this does not imply a constant fluid temperature: the latter may vary from "point" to "point" in the mixture but in any mesoscale volume the mean fluid temperature stays always equal to the mean particle temperature. Moreover, $\tilde{\rho}_f^0$ has negligible inhomogeneities around a particle because any *relative* velocity between the fluid and the particle is always much lower that the velocity of sound in the fluid. Again, this does not imply that $\tilde{\rho}_f^0$ is constant throughout the mixture because we did not make any assumption concerning the magnitude of the fluid velocity itself.

If the inhomogeneities or spatial fluctuations of $\tilde{\rho}_f^0$ and $\tilde{T}_f$ are negligible, no product of fluctuations appears in the relations [1] to [3]. The averaging procedure in a mesoscale volume is then quite simple and gives:

$$\mathrm{d}(\rho_f \epsilon_f) = \mu_f \,\mathrm{d}\rho_f + T_f \,\mathrm{d}(\rho_f s_f) - p_f \,\mathrm{d}\alpha_f, \qquad [4]$$

$$\alpha_f \,\mathrm{d}p_f = \rho_f \,\mathrm{d}\mu_f + \rho_f s_f \,\mathrm{d}T_f, \qquad [5]$$

$$\alpha_f p_f + \rho_f \epsilon_f = \rho_f \mu_f + \rho_f s_f T_f. \qquad [6]$$

The above formulas are written with the standard variables of the thermodynamics of mixtures, where instead of ρ_f^0 one rather uses the mass of fluid per unit volume of the mixture, i.e.

$$\rho_f = \alpha_f \rho_f^0, \qquad [7]$$

with α_f the fluid volume fraction.

2.2 *The thermodynamic relations for solid particles undergoing homogeneous deformations*

The local thermodynamic relations of a solid involve the local strain tensor. For a small solid particle the strain is nearly constant all over the particle (homogeneous deformation) and the trace of the strain tensor is associated with a volume change while the traceless part is associated with a change of shape. For a nearly spherical particle the shape is conveniently described by a traceless symmetric tensor C_{ij} (Frankel & Acrivos 1970) which is defined

in figure 1. With the use of C_{ij}, the Gibbs equation for the unit mass of such a particle can be written as

$$d\epsilon_s = T_s\, ds_s - p_s\, d\left(\frac{1}{\rho_s^0}\right) - \frac{\sigma_{ij}^s}{\rho_s^0}\, dC_{ij} \qquad [8]$$

where all quantities have the same meaning as for the fluid except p_s (the "particle pressure") which is in fact related to the trace of the particle stress tensor, and σ_{ij}^s which is the traceless part of this tensor. Roughly speaking, p_s is related to the Young modulus of the particle and σ_{ij}^s to its shear modulus. All the quantities in [8] appear without tildes because due to the assumption of homogeneous deformations, they are all constant throughout any given particle.

The average of the above relation in a mesoscale volume containing many particles is straightforward and leads to:

$$d(\rho_s\epsilon_s) = \mu_s\, d\rho_s + T_s\, d(\rho_s s_s) - p_s\, d\alpha_s - \alpha_s\sigma_{ij}^s\, dC_{ij}, \qquad [9]$$

where

$$\rho_s = \alpha_s\rho_s^0, \qquad [10]$$

and

$$\alpha_s = 1 - \alpha_f \qquad [11]$$

is the particle volume fraction. The average Gibbs-Duhem relation is found in a similar way and one gets

$$\alpha_s\, dp_s = \rho_s\, d\mu_s + \rho_s s_s\, dT_s + \alpha_s\sigma_{ij}^s\, dC_{ij}, \qquad [12]$$

with

$$\alpha_s p_s + \rho_s\epsilon_s = \rho_s\mu_s + \rho_s s_s T_s. \qquad [13]$$

According to what was said above concerning the temperature fluctuations, we must add to the preceeding thermodynamic relations, the following condition on the average temperatures:

$$T_f = T_s = T \qquad [14]$$

and we henceforth speak of T as the mixture temperature.

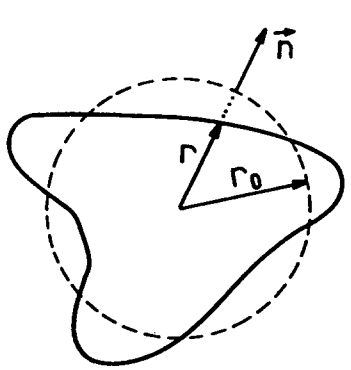

Figure 1. Definition of C_{ij}: the distance to the sphere center of a point of the deformed surface in the direction $\vec{n}$ is given by the series expansions: $r(\vec{n}) = r_0\,(1 + C_{ij}\, n_i\, n_j + \cdots)$. When the symmetric tensor C_{ij} is traceless, the volume of the deformed body is that of the sphere of radius r_0.

2.3 *The kinetic energy of the mixture*

Since the fluctuations of $\tilde{\rho}_f^0$ and $\tilde{\rho}_s^0$ are negligible (due to the low relative velocities and homogeneous deformations respectively) the velocity fluctuations only can play a role in the kinetic energy. In a mixture with two components moving at two different mean velocities $\vec{V}_s$ and $\vec{V}_f$, the kinetic energy per unit volume is thus:

$$U_{\text{kin}} = \tfrac{1}{2}\rho_s(V_s^2 + \langle(\Delta v_s)^2\rangle_s) + \tfrac{1}{2}\rho_f(V_f^2 + \langle(\Delta v_f)^2\rangle_f) \qquad [15]$$

where Δv_k is the velocity fluctuation around the mean value V_k and where $\langle\ \rangle_k$ stands for the average value in the pure k phase.

In this paper we shall suppose that the particles are rigid this means that we neglect the deformation and velocity of deformation resulting from the relative flow. We also suppose that the particle motion is not too chaotic. These two assumptions imply that we neglect the spatial and time fluctuations of the particles velocity field, hence

$$\langle(\Delta v_s)^2\rangle_s \simeq 0. \qquad [16]$$

As to the fluid, we suppose that it is not turbulent but we notice that the fluid velocity field is inhomogeneous all around the particles. For rigid particles, these spatial velocity fluctuations are a consequence of the mean relative velocity

$$\vec{W} = \vec{V}_s - \vec{V}_f, \qquad [17]$$

and they depend on the particle concentration. For spherical particles, isotropy leads to:

$$\langle(\Delta v_f)^2\rangle_f = f(\alpha_s)W^2, \qquad [18]$$

where the function $f(\alpha_s)$ is unknown except in the dilute limit where:

$$\lim_{\alpha_s\to 0} f(\alpha_s) = \alpha_s/2 \qquad [19]$$

to agree with single particle results (e.g. Landau & Lifshitz 1971). As a matter of fact, the ratio $f(\alpha_s)/\alpha_s$ gives an adimensional measure of the variation of the virtual (or added) mass with the particle concentration. If the particles, although rigid, slightly depart from a spherical shape, [18] must be amended into (Batchelor 1967):

$$\langle(\Delta v_f)^2\rangle_f = f(\alpha_s)W^2 - h(\alpha_s)C_{ij}W_iW_j \qquad [20]$$

where C_{ij} is the shape tensor introduced in the preceding section while $h(\alpha_s)$ is a second phenomenological function which, as we shall later see, is linked to the stress inside the particles.

With [16] and [20], the kinetic energy [15] is finally rewritten as

$$U_{\text{kin}} = \tfrac{1}{2}\rho V^2 + \tfrac{1}{2}\rho_{ij}W_iW_j, \qquad [21]$$

with the following classical definitions:

$$\rho = \rho_s + \rho_f,$$
$$\rho\vec{V} = \rho_s\vec{V}_s + \rho_f\vec{V}_f$$

and the introduction of the inertia tensor of the relative motion

$$\rho_{ij} = \rho^*\delta_{ij} - \rho_f h(\alpha_s)C_{ij}, \qquad [22]$$

where

$$\rho^* = \frac{\rho_f \rho_s}{\rho} + \rho_f f(\alpha_s). \qquad [23]$$

For an assembly of rigid spheres, [21] merely becomes

$$U_{\text{kin}} = \tfrac{1}{2}\rho V^2 + \tfrac{1}{2}\rho^* W^2. \qquad [24]$$

2.4 *The Gibbs equation for the mixture*

If for simplicity we neglect any energy or entropy of "mixing" (thus disregarding surface tension, brownian motion and direct interaction between the particles, cf paragraph 6) then the internal energy and entropy of the mixture are:

$$U_0 = \rho_s \epsilon_s + \rho_f \epsilon_f \qquad [25]$$

and

$$S = \rho_s s_s + \rho_f s_f. \qquad [26]$$

We are mainly concerned with the total energy

$$U = U_0 + U_{\text{kin}} \qquad [27]$$

and we want to establish the Gibbs equation of the mixture, i.e. the differential form of U as a function of the state variables. Upon inspection of [4], [9], [11] and [21], one sees that the state variables for a mixture of deformable particles in a fluid are:

$$\rho_s, \rho_f, S, \rho V_i, \rho_{ij} W_j, \alpha_s \text{ and } C_{ij}.$$

However if we restrict ourselves to the idealized case of perfectly rigid particles, the tensor C_{ij}, being a constant, is no longer a state variable and α_s being equal to ρ_s up to a constant (see [10]) is no longer an independent variable. We thus conclude that the approximation of rigid particles involves the fulfilment of two constraints:

$$\frac{\partial U}{\partial \alpha_s} = 0 \quad \text{and} \quad \frac{\partial U}{\partial C_{ij}} = 0. \qquad [28]$$

For rigid and spherical particles, the explicit form of these constraints is obtained by calculating first the complete derivatives [28], and then setting $C_{ij} = 0$. We thus get the particle stresses:

$$p_s = p_f - \frac{W^2}{2}\rho_f \frac{\mathrm{d}f}{\mathrm{d}\alpha_s}, \qquad [29]$$

and

$$\alpha_s \sigma^s_{ij} = \rho_f h(\alpha_s)\left(W_i W_j - \frac{W^2}{3}\delta_{ij}\right). \qquad [30]$$

If we compare with the single particle result for the internal stress (Biesheuvel & van Wijngaarden 1984) we deduce:

$$\lim_{\alpha_s \to 0} h(\alpha_s) = \frac{9}{20}\alpha_s. \qquad [31]$$

Moreover the single particle result for the particle pressure (van Wijngaarden 1976a) is indeed recovered with [29] and [19].

With the constraints [28], the Gibbs equation for a mixture of rigid and spherical particles is then:

$$\mathrm{d}U = \mu_s^* \,\mathrm{d}\rho_s + \mu_f^* \,\mathrm{d}\rho_f + T\,\mathrm{d}S + \vec{W} \cdot \mathrm{d}\rho^* \vec{W} + \vec{V} \cdot \mathrm{d}\rho \vec{V}, \qquad [32]$$

with

$$\mu_k^* = \mu_k - \frac{V^2}{2} - \frac{W^2}{2}\frac{\partial \rho^*}{\partial \rho_k} \quad (k = f \text{ or } s). \qquad [33]$$

It differs from the usual Gibbs equation (De Groot & Mazur, 1969) by the terms involving the relative velocity, and above all through the appearance of the invariant (or internal) momentum $\rho^* \vec{W} = \partial U / \partial \vec{W}$ as a new state variable. That this momentum is a more convenient variable than $\vec{W}$ itself was already suggested in Landau & Lifshitz (1971) from a study of the motion of a single particle in a uniform flow; this result was later extended to more general flows by Lhuillier (1982). One may notice that it was necessary to deal for a while with deformable particles (in [9] for instance) to get the relevant constraints [29] and [30] for rigid particles. It is also possible to recover the above constraints with arguments of virtual deformation.

2.5 *The equations of motion and the entropy production*

A thermodynamic state of the mixture being defined by the knowledge of ρ_s, ρ_f, U, $\rho\vec{V}$ and $\rho^*\vec{W}$, to describe the evolution in time of a state we must have equations for the evolution of each variable.

Galilean invariance is satisfied with:

$$\frac{\partial \rho_s}{\partial t} + \vec{\nabla} \cdot (\rho_s \vec{V} + \vec{J}) = 0 \qquad [34]$$

$$\frac{\partial \rho_f}{\partial t} + \vec{\nabla} \cdot (\rho_f \vec{V} - \vec{J}) = 0 \qquad [35]$$

$$\frac{\partial \rho \vec{V}}{\partial t} + \vec{\nabla} \cdot (\rho \overline{\vec{V}\vec{V}} + \bar{\pi}) = \rho \vec{g} \qquad [36]$$

$$\pi_{ij} = \pi_{ji}$$

$$\frac{\partial U}{\partial t} + \vec{\nabla} \cdot (U\vec{V} + \vec{V} \cdot \bar{\pi} + \vec{Q}) = \rho \vec{g} \cdot \vec{V} \qquad [37]$$

$$\frac{\partial \rho^* \vec{W}}{\partial t} + \vec{\nabla} \cdot (\rho^* \overline{\vec{W}\vec{V}} + \bar{R}) = \vec{F} \qquad [38]$$

and although α_s is no more a state variable in the case of rigid particles it will prove useful to know its evolution equation deduced from [10] and [34]:

$$\frac{\partial \alpha_s}{\partial t} + \vec{\nabla} \cdot \left(\alpha_s \vec{V} + \frac{\vec{J}}{\rho_s^0}\right) = 0. \qquad [39]$$

In the above equation, $\vec{J}$ is the mass flux, $\bar{\bar{\pi}}$ the momentum flux, $\vec{Q}$ the heat flux, $\bar{\bar{R}}$ the flux of the internal momentum and $\vec{F}$ its source. Their explicit expressions are yet unknown.

From the Gibbs equation [32] and the evolution equations [34] to [38] it is not too difficult to deduce the entropy balance equation

$$\frac{\partial S}{\partial t} + \vec{\nabla} \cdot (S\vec{V} + \vec{S}) = -\frac{1}{T}[\vec{S} \cdot \vec{\nabla} T + \vec{J} \cdot \vec{\nabla}(\mu_s^* - \mu_f^*) + \vec{W} \cdot \vec{F} + (\bar{\bar{\pi}} - p\bar{\bar{I}}) : \overline{\overline{\nabla V}} + \bar{\bar{R}} : \overline{\overline{\nabla V}}], \quad [40]$$

where the entropy flux $\vec{S}$ is related to the other fluxes by

$$T\vec{S} = \vec{Q} - (\mu_s^* - \mu_f^*)\vec{J} - \vec{W} \cdot \bar{\bar{R}}, \quad [41]$$

and the scalar p, defined as the sum

$$p = -U + \rho_f \mu_f^* + \rho_s \mu_s^* + TS + \rho^* W^2 + \rho V^2,$$

will now be proved to be the mean pressure of the mixture. Using [6], [13], [24], and [33] we can rewrite p as:

$$p = \alpha_s \rho_s + \alpha_f \rho_f + \frac{W^2}{2}\left(\rho^* - \rho_f \frac{\partial \rho^*}{\partial \rho_f} - \rho_s \frac{\partial \rho^*}{\partial \rho_s}\right).$$

But any acceptable expression for ρ^* must be extensive in ρ_s and ρ_f, hence the requirement

$$\rho^*(\rho_s, \rho_f, \alpha_s) = \rho_f \frac{\partial \rho^*}{\partial \rho_f} + \rho_s \frac{\partial \rho^*}{\partial \rho_s}. \quad [42]$$

Such a condition is obviously satisfied by [23] and the pressure p is thus nothing but the mean pressure which, with [29], becomes

$$p = \alpha_s p_s + \alpha_f p_f = p_f - \frac{W^2}{2} \alpha_s \rho_f \frac{\mathrm{d}f}{\mathrm{d}\alpha_s}, \quad [43]$$

in agreement with what was found by van Wijngaarden (1976a) in the dilute limit.

At this point, the usual procedure would be to deduce from [40] and the methods of irreversible thermodynamics, the most general expressions of the fluxes $\vec{S}$, $\vec{J}$, $\vec{F}$, $\bar{\bar{\pi}}$ and $\bar{\bar{R}}$ compatible with a positive entropy production. Each of these fluxes is in fact made of a reversible and a irreversible part. For the irreversible part, no doubt that irreversible thermodynamics is a powerful tool. But to determine the non-dissipative contributions (the ones which do not play a role in the entropy production) the problem is left open because we have a lot of different possibilities and physical arguments are necessary if one does not want to cover pages with general but tremendous formulas. In fact to solve that problem of the non-dissipative parts of the fluxes we propose in this paper to compare the "thermodynamic" equations [34] to [38] with the corresponding average equations that can be found in standard text-books (Ishii 1975) and articles (Nigmatulin 1979, Drew 1983). This procedure already proved helpful in the case of a solution of deformable polymers (Lhuillier 1983).

3. THE NON-DISSIPATIVE PART (OR $\vec{W}$-DEPENDENCE) OF THE FLUXES

3.1 *The mass flux*

If we compare [34] and [35] to the standard form of the average equation of mass conservation

$$\frac{\partial \rho_k}{\partial t} + \vec{\nabla} \cdot \rho_k \vec{V}_k = 0 \qquad (k = f \text{ or } s)$$

we deduce that:

$$\vec{J} = \frac{\rho_f \rho_s}{\rho} \vec{W}. \qquad [44]$$

With this result in mind, [38] is somewhat reminiscent of the extended thermodynamics scheme (Müller 1984) where the flux $\vec{J}$ is considered as a state variable and obeys an equation of motion instead of being merely proportional to the gradient of chemical potential as in classical thermodynamics. And in fact, if it were not for the added mass term represented by $f(\alpha_s)$ in [23] we would have $\vec{J} \equiv \rho^* \vec{W}$ and [38] would be exactly the equation written in the frame of extended thermodynamics.

3.2 *The entropy flux*

Since the entropy of the particles (resp. fluid) is convected with the particles (resp. fluid) velocity, the total entropy flux must be written as:

$$\rho_s s_s \vec{V}_s + \rho_f s_f \vec{V}_f + \vec{s}.$$

Comparing with [40], we deduce that we must write:

$$\vec{S} = \frac{\rho_f \rho_s}{\rho} (s_s - s_f) \vec{W} + \vec{s}. \qquad [45]$$

Obviously the above result is to be modified when some entropy of mixing is present (due for instance to a temperature-dependent surface tension, or to a configuration entropy of the particles in the fluid, cf section 6). In that case one must determine the velocity of convection of this extra entropy and it is this new contribution to $\vec{S}$ which is at the origin of a Soret effect in the mixture (cf Lhuillier (1983) for an example concerning polymer solutions).

3.3 *The momentum flux*

The general form of the average momentum flux (Ishii 1975) is for a mixture of fluid and solid particles

$$\rho_f \overline{V_f V_f} + \rho_s \overline{V_s V_s} + \rho_f \langle \overline{\Delta v_f \Delta v_f} \rangle_f + \rho_s \langle \overline{\Delta v_s \Delta v_s} \rangle_s + \alpha_f p_f \bar{I} + \alpha_s (p_s \bar{I} + \bar{\mathrm{O}}_s^-) + \bar{\mathrm{T}},$$

where the elastic stress inside the particles has been taken into account and $\bar{\mathrm{T}}$ represents the average viscous contribution. Remembering what was said in the section on kinetic energy about the rigidity of the particles and their not too chaotic motion, we have (cf [16])

$$\langle \overline{\Delta v_s \Delta v_s} \rangle_s \simeq 0.$$

As to the "Reynolds stress" associated with the spatial fluctuations of the fluid velocity we must propose a general expression compatible with [18], and this is achieved by:

$$\langle \overline{\Delta v_f \Delta v_f} \rangle_f = f(\alpha_s) \overline{WW} + g(\alpha_s) \left(\frac{W^2}{3} \bar{I} - \overline{WW} \right) \qquad [46]$$

where $g(\alpha_s)$ is a third (and last !) unknown function of the particle concentration. Comparison with dilute solution results (Nigmatulin 1978) leads to:

$$\lim_{\alpha_s \to 0} g(\alpha_s) = \frac{9}{20}\alpha_s \qquad [47]$$

with the important consequence that the dilute limits of both g and h are equal (cf [31]). If we now remember the definition [43] of the mean pressure and the expression [30] for the particle stress, we see that the average momentum flux is finally

$$p\bar{I} + \rho_f\overline{\overline{V_fV_f}} + \rho_s\overline{\overline{V_sV_s}} + \rho_f f\,\overline{\overline{WW}} + \rho_f(g-h)\left(\frac{W^2}{3}\bar{I} - \overline{\overline{WW}}\right) + \bar{\bar{\mathrm{T}}},$$

and comparing with [36] we conclude that we must write $\bar{\bar{\pi}}$ as

$$\bar{\bar{\pi}} = p\bar{I} + \rho^*\overline{\overline{WW}} + \rho_f(g-h)\left(\frac{W^2}{3}\bar{I} - \overline{\overline{WW}}\right) + \bar{\bar{\mathrm{T}}} \qquad [48]$$

So, two scalar functions linked to the velocity fluctuations and a third one linked to the particle stress appear in the expression of the momentum flux.

3.4 *The energy flux*

The total average energy flux (Ishii 1975) can be written as

$$\alpha_f\langle(\rho_f^0\epsilon_f + \tilde{p}_f + \tfrac{1}{2}\rho_f^0 v_f^2)\vec{v}_f\rangle_f + \alpha_s\langle(\rho_s^0\epsilon_s + \tilde{p}_s + \tfrac{1}{2}\rho_s^0 v_s^2)\vec{v}_s + \bar{\bar{\sigma}}^s\cdot\vec{v}_s\rangle_s$$
$$+ \text{viscous contribution} + \text{heat conductivity contribution}.$$

In conformity we our previous hypothesis we neglect all fluctuations inside the particles and in their translational motion but we take into account the velocity and pressure fluctuations inside the fluid through the use of the result [A.4] established in appendix I. We rearrange the terms so as to present the total energy flux like in [37], and we then find that we must write the flux $\vec{Q}$ as:

$$\vec{Q} = \frac{\rho_f\rho_s}{\rho}\vec{W}\left[\epsilon_s + \frac{p_s}{\rho_s^0} + \frac{1}{2}(V_s - V)^2 - \epsilon_f - \frac{p_f}{\rho_f^0} - \frac{1}{2}(V_f - V)^2 - \frac{1}{2}\langle\Delta v_f^2\rangle_f\right]$$
$$+ \frac{\rho_f}{\rho}\vec{W}\cdot[\alpha_s\bar{\bar{\sigma}}^s + \rho_f\langle\overline{\overline{\Delta v_f\Delta v_f}}\rangle_f]\Big] + \vec{q}.$$

Taking into account the thermodynamic relations [6] and [13], the expressions [46] and [30] for the velocity fluctuations and the particle stress, and finally the definition [33] of μ_k^* we can rewrite the above expression as:

$$\vec{Q} = \frac{\rho_f\rho_s}{\rho}\vec{W}\,[\mu_s^* - \mu_f^* + T(s_s - s_f)]$$
$$+ \vec{W}\cdot\left[\rho^*\frac{\rho_f - \rho_s}{\rho}\overline{\overline{WW}} + \frac{\rho_f^2}{\rho}(g-h)\left(\frac{W^2}{3}\bar{I} - \overline{\overline{WW}}\right)\right] + \vec{q}. \qquad [49]$$

3.5 *The internal momentum flux*

The internal momentum flux is not independent of the above-mentioned fluxes since we have the thermodynamic relation [41] which links $\bar{\bar{R}}$ to $\vec{S}$, $\vec{Q}$ and $\vec{J}$. The result [49] for the

heat flux $\vec{Q}$ is quite compatible with such a relation provided $\overline{\overline{R}}$ is given by:

$$\overline{\overline{R}} = \rho^* \frac{\rho_f - \rho_s}{\rho} \overline{\overline{WW}} + \frac{\rho_f^2}{\rho}(g - h)\left(\frac{W^2}{3}\overline{\overline{I}} - \overline{\overline{WW}}\right) + \overline{\overline{r}} \qquad [50]$$

and the dissipative part of the fluxes are linked by the relation

$$\vec{q} = T\vec{s} + \overline{\overline{W}} \cdot \overline{\overline{r}}. \qquad [51]$$

We now have reached our goal which was to distinguish in the various fluxes, the dissipative and the non-dissipative (or $\vec{W}$-dependent) parts. The results [44] and [45] are rather evident. The result [48] supposes a regular or "laminar" flow of particles in the fluid and results [49] and [50] rest on the special pressure-velocity correlation [A.4] which was established for ideal fluids.

4. THE INTERNAL MOMENTUM FORCE

With the expressions [44], [45], [48] and [50] for the fluxes the entropy balance law [40] is transformed into

$$\begin{aligned}\frac{\partial S}{\partial t} + \vec{\nabla} \cdot (S\vec{V} + \vec{S}) = -\frac{1}{T}\Bigg[&\vec{s} \cdot \vec{\nabla} T + \vec{F} \cdot \vec{W} + \overline{\overline{r}} : \overline{\overline{\nabla W}} + \overline{\overline{\mathrm{T}}} : \overline{\overline{\nabla V}} \\ &+ \frac{\rho_f \rho_s}{\rho}(s_s - s_f)\,\vec{W} \cdot \vec{\nabla} T + \frac{\rho_f \rho_s}{\rho}\vec{W} \cdot \vec{\nabla}(\mu_s^* - \mu_f^*) + \rho^* \overline{\overline{WW}} : \overline{\overline{\nabla V}} \\ &+ \rho^* \frac{\rho_f - \rho_s}{\rho}\overline{\overline{WW}} : \overline{\overline{\nabla W}} + \rho_f(g - h)\left(\frac{W^2}{3}\overline{\overline{I}} - \overline{\overline{WW}}\right) : \left(\overline{\overline{\nabla V_s}} - \overline{\overline{W \nabla \frac{\rho_f}{\rho}}}\right)\Bigg].\end{aligned}$$

In the entropy production (right-hand side of the above equation) all the terms coming from the $\vec{W}$-dependence of the fluxes have an odd parity in a time-reversal. We conclude that these terms must disappear since an entropy production being irreversible, cannot change in a time-reversal. There is only one way to fulfill this requirement and this amounts to write $\vec{F}$ in the form

$$\begin{aligned}\vec{F} = \vec{f} &- \frac{\rho_f \rho_s}{\rho}\left(\frac{1}{\rho_s^0}\vec{\nabla} p_s - \frac{1}{\rho_f^0}\vec{\nabla} p_f\right) - \frac{\rho_f \rho_s}{\rho}\left(\frac{\partial \rho^*}{\partial \rho_f} - \frac{\partial \rho^*}{\partial \rho_s}\right)(\vec{W} \cdot \vec{\nabla})\,\vec{W} \\ &- \frac{\rho_f \rho_s}{\rho}\frac{\vec{W}}{2}\,\vec{W} \cdot \vec{\nabla}\left(\frac{\partial \rho^*}{\partial \rho_f} - \frac{\partial \rho^*}{\partial \rho_s}\right) - \rho^*(\vec{W} \cdot \vec{\nabla})\,\vec{V} \\ &- \rho^* \frac{\rho_f - \rho_s}{\rho}(\vec{W} \cdot \vec{\nabla})\,\vec{W} + \rho_f(g - h)\left(\frac{W^2}{3}\overline{\overline{I}} - \overline{\overline{WW}}\right) \cdot \vec{\nabla}\frac{\rho_f}{\rho} \\ &+ \rho_f(g - h)\,((\vec{W} \cdot \vec{\nabla})\,\vec{V}_s) - \left(\frac{\vec{W}}{3}\vec{\nabla} \cdot \vec{V}_s\right) + \text{terms orthogonal to } \vec{W}.\end{aligned} \qquad [52]$$

To get that expression we used the following identity:

$$\vec{\nabla}(\mu_s - \mu_f) + (s_s - s_f)\,\vec{\nabla} T = \frac{1}{\rho_s^0}\vec{\nabla} p_s - \frac{1}{\rho_f^0}\vec{\nabla} p_f,$$

which was derived from [5] and [12]. It must be noticed that there is a certain arbitrariness in writing [52] since this expression is only defined up to terms orthogonal to $\vec{W}$. In fact we chose the form which is the most convenient for the subsequent calculations. With that expression the entropy balance law merely becomes

$$\frac{\partial S}{\partial t} + \vec{\nabla} \cdot (S\vec{V} + \vec{S}) = -\frac{1}{T}[\vec{s} \cdot \vec{\nabla} T + \vec{f} \cdot \vec{W} + \overline{\overline{r}} : \overline{\overline{\nabla W}} + \overline{\overline{\mathrm{T}}} : \overline{\overline{\nabla V}}] \qquad [53]$$

and the expressions of $\vec{s}$, $\vec{f}$, $\bar{r}$ and $\bar{\bar{T}}$ follow from the usual methods of irreversible thermodynamics (De Groot & Mazur 1969). In the dissipative entropy flux $\vec{s}$ will appear the thermal conductivity of the mixture, and $\vec{f}$ will represent some generalization of the Stokes's force to arbitrary concentrations of particles. No dissipative coupling is expected between $\vec{f}$ and $\vec{s}$ since $\vec{W}$ and $\vec{\nabla} T$ behave differently in a time-reversal (however, a nondissipative coupling was yet taken into account in [45] and [52] through the $\vec{W}$-dependence of $\vec{S}$ and $\vec{\nabla} T$ dependence of $\vec{F}$). A more original feature is perhaps the coupling between $\bar{r}$ and $\bar{\bar{T}}$ with which we may expect to derive some kind of generalized Faxen's force from $\vec{\nabla} \cdot \bar{r}$ (Lhuillier 1983). Moreover it is possible for $\bar{r}$ to have an antisymmetric part proportional to $\vec{\nabla} \times \vec{W}$. Lastly one sees that the above entropy production no longer involves the usual gradient of chemical potential (compare with [40]). This is because the mass flux is proportional to $\vec{W}$ and there is an equation of evolution for $\rho^* \vec{W}$. If inertia terms are suppressed from this equation, one recovers the classical Fick's law, as was first established in the pioneering work of Müller (1968). But we stop here these comments since we are mainly interested in non-dissipative, i.e. inertia terms.

5. THE INTERPHASE MOMENTUM EXCHANGE

The equation [38] for $\rho^* \vec{W}$ is complemented by [50] and [52] for $\bar{\bar{R}}$ and $\vec{F}$. Together with [36] and [48] for $\bar{\bar{\pi}}$, this achieves the dynamical description of the mixture. However, in the multiphase flow community, the variables $\vec{V}_s$ and $\vec{V}_f$ are far more usual than $\rho\vec{V}$ and $\rho^* \vec{W}$. For this reason we now transform [38] with [50] and [52] into:

$$\begin{aligned}
\rho^* \left(\frac{D_s\vec{V}_s}{Dt} - \frac{D_f\vec{V}_f}{Dt}\right) = & -\frac{\rho_f\rho_s}{\rho}\left(\frac{1}{\rho_s^0}\vec{\nabla} p_s - \frac{1}{\rho_f^0}\vec{\nabla} p_f\right) - \alpha_s \vec{\nabla}(p_f - p_s) \\
& - \frac{\rho_s}{\rho}(p_f - p_s)\vec{\nabla}\alpha_s \\
& - \frac{\rho_f}{\rho}\vec{\nabla}\cdot\left[\rho_f(g-h)\left(\frac{W^2}{3}\bar{I} - \overline{\overline{WW}}\right) + \rho_f f\,\overline{\overline{WW}}\right] \\
& + \vec{\nabla}\cdot\left(\alpha_s\frac{\partial\rho^*}{\partial\alpha_s}\overline{\overline{WW}}\right) - \alpha_s\frac{W^2}{2}\vec{\nabla}\frac{\partial\rho^*}{\partial\alpha_s} \\
& + \rho_f(g-h)\left((\vec{W}\cdot\vec{\nabla})\vec{V}_s - \frac{\vec{W}}{3}\vec{\nabla}\cdot\vec{V}_s\right) + \alpha_s\frac{\partial\rho^*}{\partial\alpha_s}\vec{W}(\vec{\nabla}\cdot\vec{V}_f) + \vec{f} \\
& - \vec{\nabla}\cdot\bar{r} + \text{terms orthogonal to } \vec{W}.
\end{aligned} \qquad [54]$$

What is the interphase momentum force $\vec{F}_{\text{inter}}$ that follows from the above result? If we discard for a while the dissipative terms and remember that we neglected from the outset the Reynolds stress of the solid particles, the usual way to write the averaged momentum equation is:

$$\rho_s\frac{D_s\vec{V}_s}{Dt} + \vec{\nabla}\alpha_s p_s + \vec{\nabla}\cdot\alpha_s\bar{\bar{\sigma}}^s = \rho_s\vec{g} + \vec{F}_{\text{inter}} \qquad [55]$$

$$\rho_f\frac{D_f\vec{V}_f}{Dt} + \vec{\nabla}\alpha_f p_f + \vec{\nabla}\cdot\rho_f\langle\overline{\Delta v_f\Delta v_f}\rangle_f = \rho_f\vec{g} - \vec{F}_{\text{inter}} \qquad [56]$$

from which one finds

$$\begin{aligned}
\frac{\rho_f\rho_s}{\rho}\left(\frac{D_s\vec{V}_s}{Dt} - \frac{D_f\vec{V}_f}{Dt}\right) = \vec{F}_{\text{inter}} - \frac{\rho_f\rho_s}{\rho}\left(\frac{1}{\rho_s^0}\vec{\nabla} p_s - \frac{1}{\rho_f^0}\vec{\nabla} p_f\right) - \frac{\rho_s}{\rho}(p_f - p_s)\vec{\nabla}\alpha_s \\
- p_s\vec{\nabla}\alpha_s - \frac{\rho_f}{\rho}\vec{\nabla}\cdot\alpha_s\bar{\bar{\sigma}}^s + \frac{\rho_s}{\rho}\vec{\nabla}\cdot\rho_f\langle\overline{\Delta v_f\Delta v_f}\rangle_f.
\end{aligned}$$

Comparing this equation with [54] we finally arrive at the following interphase force

$$\begin{aligned}\vec{F}_{\text{inter}} = & -\alpha_s \vec{\nabla} p_f - \rho_f f\left(\frac{D_s\vec{V}_s}{Dt} - \frac{D_f\vec{V}_f}{Dt}\right) + \vec{\nabla}\,\alpha_s p_s \\ & - \vec{\nabla}\cdot\left[\rho_f g\left(\frac{W^2}{3}\bar{\bar{I}} - \overline{\overline{WW}}\right) + \rho_f\left(f - \alpha_s\frac{\mathrm{d}f}{\mathrm{d}\alpha_s}\right)\overline{\overline{WW}}\right] + \rho_f\alpha_s\frac{\mathrm{d}f}{\mathrm{d}\alpha_s}\vec{W}\,(\vec{\nabla}\cdot\vec{V}_f) \\ & + \rho_f(g-h)\left((\vec{W}\cdot\vec{\nabla})\,\vec{V}_s - \frac{\vec{W}}{3}\vec{\nabla}\cdot\vec{V}_s\right) - \alpha_s\frac{W^2}{2}\vec{\nabla}\left(\rho_f\frac{\mathrm{d}f}{\mathrm{d}\alpha_s}\right) \\ & + \text{terms orthogonal to } \vec{W} + \text{dissipative terms.}\end{aligned} \qquad [57]$$

The first two terms on the right-hand side of [57] were expected and represent respectively the Archimede's force and the virtual mass force. The third and fourth terms are in the form of a gradient and a divergence. That the interphase force should involve such terms was already suggested by Nigmatulin (1979) on the ground of a contribution of the particles intersected by any mesoscale volume. The dilute limit of these two terms agrees with the single particle calculations of Nigmatulin (1978) when [29], [19] and [47] are taken into account, thus giving some confidence in the way we got [57] for $\vec{F}_{\text{inter}}$. The fifth to seventh terms of [57] are new and we cannot yet propose simple physical interpretations for them. As to the undetermined forces orthogonal to $\vec{W}$, we do not have many possibilities, except perhaps the Saffman's force (1965). It must be noted that a $\vec{W}\times(\vec{\nabla}\times\vec{V}_f)$ term was found by Drew (1983) when forcing the interphase force to fulfill the principle of material frame indifference or objectivity. But objectivity is a thorny matter when inertia effects are involved (Ryskin & Rallison 1980, Bampi & Morro 1981, Lumley 1983). If we suppress all terms orthogonal to $\vec{W}$, then our expression of $\vec{F}_{\text{inter}}$ is certainly not compatible with the objectivity principle, due to the virtual mass term. However in [38] from which we started, both $\overline{\overline{R}}$ and $\vec{F}$ were objective. So we have the feeling that with a change of variables (here the change of $\vec{V}_s$ and $\vec{V}_f$ into $\rho\vec{V}$ and $\rho^*\vec{W}$; also see Woods (1983) for another example) it is always possible to have objective constitutive relations; but it must be stressed that in contrast with the constitutive relations, the equations of evolution like [38] are definitely not frame indifferent.

6. THE PROBLEM OF CHARACTERISTICS

Since a few years, it has been customary to examine the mathematical features of the equations proposed to describe two-phase flows. In particular, much has been written concerning the characteristics of these equations in connection with the well-posedness of the two phase flow problems. Although it was shown that in some cases complex characteristics are not incompatible with well-posedness (Bouré & Latrobe 1976, Klebanov *et al.* 1982) we would be glad if our set of equations happened to have only real characteristic directions.

Eliminating the fluid pressure from [55] and [56] we get an equation involving the velocities only, the one-dimensional form of which is (compare with Van Beek 1982):

$$A\left(\frac{\partial V_f}{\partial t} + V_f\frac{\partial V_f}{\partial x}\right) - B\left(\frac{\partial V_s}{\partial t} + V_s\frac{\partial V_s}{\partial x}\right) + \frac{aW^2}{\alpha_s}\frac{\partial\alpha_s}{\partial x} + 2bW\frac{\partial V_s}{\partial x} - cW\frac{\partial V_f}{\partial x} = 0 \qquad [58]$$

where

$$A = 2\left(1 + \frac{f}{\alpha_s}\right)$$

$$B = 2\left(\frac{\rho_s^0}{\rho_f^0} + \frac{f}{\alpha_s}\right)$$

$$a = \frac{4}{3}\alpha_f \frac{\mathrm{d}}{\mathrm{d}\alpha_s}(g - h) - \frac{4}{3}(g - h) + 2f + \alpha_f\alpha_s \frac{\mathrm{d}^2 f}{\mathrm{d}\alpha_s^2}$$

$$b = \frac{2}{3}(1 + 2\alpha_f)\frac{g - h}{\alpha_s} + (1 + \alpha_s)\frac{\mathrm{d}f}{\mathrm{d}\alpha_s} - 2\alpha_f \frac{f}{\alpha_s}$$

$$c = \frac{2\alpha_f}{\alpha_s}\left(\alpha_s \frac{\mathrm{d}f}{\mathrm{d}\alpha_s} - 2f + \frac{4}{3}(g - h)\right).$$

Note that all the terms orthogonal to $\vec{W}$ in [57] disappear from [58] and thus play no role in the characteristics of the equations. If we suppose for a while that the liquid is incompressible, [58] will be associated with the one-dimensional continuity equations

$$\frac{\partial \alpha_s}{\partial t} + \frac{\partial}{\partial x}\alpha_s V_s = 0, \qquad [59]$$

$$\frac{\partial \alpha_f}{\partial t} + \frac{\partial}{\partial x}\alpha_f V_f = 0. \qquad [60]$$

The characteristics of the set [58]–[60] of quasi-linear first order differential equations are bound to the roots ν of the equation

$$(\alpha_s A + \alpha_f B)(\nu - V_s)^2 + 2W(\nu - V_s)\left(\alpha_f b + \alpha_s\left(A + \frac{c}{2}\right)\right) + W^2(\alpha_f a + \alpha_s(A + c)) = 0.$$

The characteristic directions are thus all real provided:

$$\left(\alpha_f b + \alpha_s\left(A + \frac{c}{2}\right)\right)^2 > (\alpha_f a + \alpha_s(A + c))(\alpha_s A + \alpha_f B). \qquad [61]$$

This is a rather complicated inequality involving f, $g - h$ and their derivatives together with ρ_s^0/ρ_f^0 which appears in B. Let us examine it for dilute solutions. In this limit we write the series expansions of f and $g - h$ as:

$$f = k_1\alpha_s + k_2\alpha_s^2 + \ldots$$

$$g - h = K_1\alpha_s + K_2\alpha_s^2 + \ldots$$

Let us first suppose that $K_1 \neq 0$. One then finds

$$a = {}^4/_3\, K_1 + 0(\alpha_s)$$

$$b = 2K_1 + 0(\alpha_s)$$

and the condition [61] for real characteristics becomes

$$b^2 > aB$$

or

$$K_1^2 > \frac{2}{3}K_1\left(k_1 + \frac{\rho_s^0}{\rho_f^0}\right)$$

This inequality is satisfied for any *negative* K_1 (and also for a sufficiently large positive value of K_1, but depending on k_1 and ρ_s^0/ρ_f^0). It is not difficult to get convinced that a negative K_1 corresponds in [58] to a $\vec{\nabla}\alpha_s$ force in the direction opposite to the acceleration of the particles.

This was already pointed out by Van Beek (1982) and Micaelli (1982) who observed that whenever $a < 0$ in [58], the particles tend to move in the direction opposite to $\vec{\nabla}\alpha_s$, thus lowering any already existing gradient of concentration, a potentially stable situation.

Unfortunately if we take the results [31] and [47] for granted we are obliged to accept that

$$K_1 = 0,$$

In this case both a and b are proportional to α_s and the inequality [61] becomes:

$$A + c + \frac{a}{\alpha_s} < 0$$

or

$$1 + k_1 + k_2 + \tfrac{4}{3} K_2 < 0 \qquad [62]$$

If the value $k_1 = 0.5$ seems firmly established (cf [19]), the value of k_2 is still in dispute (k_2 is 1.5 for Zuber (1964), 1.66 for van Wijngaarden (1976b) and 0 for Nigmatulin (1978)) while K_2 is unknown, since Nigmatulin (1978) only found:

$$g = \frac{9}{20}\alpha_s(1 - \alpha_s) + 0(\alpha_s^3),$$

but nothing concerning h. So, we must wait for future calculations to decide if [62] is satisfied or not in dilute solutions.

It must be kept in mind that other physical phenomena can lead to a $\vec{\nabla}\alpha_s$ force and thus can change the nature of the characteristics. One can think of a direct interaction between the particles (Klebanov *et al.* 1982) but even more simply, of a configuration entropy of the particles dispersed in the fluid. By analogy with the theory of dilute solutions (De Groot & Mazur 1969) we can change [26] into:

$$S = \rho_s s_s + \rho_f s_f - \frac{k_B}{m_s}\rho_s \log\frac{\alpha_s}{\alpha_s^*},$$

where m_s is the mass per particle, k_B is the Boltzmann constant and $\alpha_s^* = 0.6$ is the upper volume fraction of the dispersion.

As a consequence, a term like

$$\Delta\mu_s = \frac{k_B T}{m_s}\log\frac{\alpha_s}{\alpha_s^*}$$

must be added to the particles chemical potential and then appears in $\vec{F}$ [52] the force

$$-\frac{\rho_f\rho_s}{\rho}\vec{\nabla}\Delta\mu_s = -\frac{\rho_f}{\rho}\frac{k_B T}{T_s}\vec{\nabla}\alpha_s$$

where T_s is the particle volume. This diffusion force, very similar to the Brownian motion force, acts in the "good" direction, but it must be stressed that its structure is very different from the inertia $\vec{\nabla}\alpha_s$ force which involves the square of the relative velocity (cf [58]). As a consequence the "Brownian" diffusion force can lead to real characteristics only if the relative velocity is low enough, while the inertia diffusion force can lead to real characteristics whatever is the relative velocity. In short, the "Brownian" diffusion force is not the whole

story and it is of utmost importance to verify if one of the inequalities $K_1 < 0$ or [62] is satisfied.

7. CONCLUSION

The main results of this paper concern the equations of motion of a dispersed solid-fluid mixture and they follow from a combination of:

(a) A rather usual thermodynamic approach represented by the Gibbs equation [32], the equations of motion [34] to [38] and the entropy balance law [40]. These are completed by [41] relating the fluxes, expression [43] for the mean pressure and expressions [29] and [30] for the particles internal stresses.

(b) A less usual approach (based on compatibility between average and thermodynamic equations) which allowed us to derive the non-dissipative parts of the fluxes $\vec{J}, \vec{S}, \bar{\bar{\pi}}, \vec{Q}, \bar{\bar{R}}$ and also of the force $\vec{F}$. These are given by [44], [45], [48], [49], [50] and [52]. The dissipative parts of the fluxes are linked by [51] and they can be deduced from the entropy production [53].

Instead of the Truesdellian rational thermodynamics transposed to multiphase mixtures by Bedford & Drumheller (1983) or Drew & Lahey (1979) for instance, we used here a thermodynamic approach "à la Landau" that was first applied to superfluid helium (cf Kalatnikov 1955 and Landau & Lifshitz 1971). At first sight these two approaches seem very different. For instance we assumed at the outset a given expression for the Gibbs equation and then, with the techniques of irreversible thermodynamics, we deduced the unknown terms in the evolution equations. Galilean invariance was the only requisite and no reference was made to the "principle of equipresence" nor to the "principle of material frame indifference." Despite this, when applied to the same material both approaches often led in the past to very similar results (Germain *et al.* 1983). We nevertheless emphasized in this paper some points that were perhaps not sufficiently appreciated up to now in the study of a two-velocity mixture.

The main point is that, when dealing with inertia effects related to the relative velocity, the best state variable is the momentum $\rho^* \vec{W} = \partial U/\partial \vec{W}$ and not $\vec{W}$ itself. Discarding this internal momentum to the benefit of $\vec{W}$ is all the more surprising than with inertia linked to rotation, everyone is accustomed to use the angular momentum $\partial U/\partial \vec{\Omega}$ and not $\vec{\Omega}$ itself. Moreover with the use of $\rho^* \vec{W}$ as a state variable, the constitutive relations are automatically frame-indifferent, despite the presence of "added mass effects."

It is true that $\rho^* \vec{W}$ and $\rho \vec{V}$ have a much less intuitive meaning than the fluid and particle velocities, but if one prefers to deal with $\vec{V}_f$ and $\vec{V}_s$, then a second important remark arises: the expression of the interphase force $\vec{F}_{\text{inter}}$ must involve a divergence term (Nigmatulin 1979) which is responsible—together with the divergence of the particle stress—for the main inertia contribution to the $\vec{\nabla}\alpha_s$ force, and consequently plays a decisive role in the nature of the characteristics. It happens that our present knowledge of the fluid velocity fluctuations and of the particle stresses in the dilute limit, is not yet sufficient to decide if our system of equations is hyperbolic or not.

We have also shown that [29] and [30] were a consequence of the elimination of α_s and C_{ij} as independant state variables. This is evidently valid for rigid particles only. The next step would be to extend the present approach to spherical particles with a variable radius (Nigmatulin 1978 and Biesheuvel & van Wijngaarden 1984 already worked in that direction) and more generally to particles suffering an homogeneous deformation. Then we could arrive at what could be called a two-velocity fluid with microstructure, a necessary generalization of Eringen's work (1968) when dealing with suspensions.

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Note: I recently received a preprint by Pr. J. A. Geurst who used a variation principle to get the equations of a two-velocity mixture. His final equations are rather similar to ours if one supposes that $g = h$ whatever α_s is. However, for his equations to have real characteristics, the added mass must decrease with the void fraction and even cancel for $\alpha_s = 1/3$. This unsatisfactory result is linked to his assumption $g = h$ and points out the importance of future calculations of $g(\alpha_s)$ and $h(\alpha_s)$.

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APPENDIX I

Pressure and velocity fluctuations in the motion of a particle in an ideal fluid

When introduced into a pre-existing flow $\vec{V}_\infty(\vec{r}, t)$ with pressure $p_\infty(\vec{r}, t)$, a particle moving along a trajectory $\vec{r}_0(t)$ with a velocity $\vec{V}_s = d\vec{r}_0/dt$ modifies the velocity and pressure fields around it into $\vec{v}_f(\vec{r}, t)$ and $\tilde{p}_f(\vec{r}, t)$. Let us suppose that the relative velocity in high enough for the flow around the particle to be that of an ideal fluid with a curl-free velocity field. Then Bernoulli's theorem implies:

$$\tilde{p}_f = p_\infty - \frac{1}{2}\rho_f^0(\vec{v}_f - \vec{V}_\infty)^2 - \rho_f^0\vec{V}_\infty \cdot (\vec{v}_f - \vec{V}_\infty) - \rho_f^0\left(\frac{\partial\phi}{\partial t} - \frac{\partial\phi_\infty}{\partial t}\right)$$

where ϕ and ϕ_∞ are the respective potentials of $\vec{v}_f$ and $\vec{V}$. Denoting by $\langle\ \rangle_f$ a mean value over a volume of fluid centered on the particle one gets the following result:

$$\langle\tilde{p}_f\vec{v}_f\rangle_f = \langle\tilde{p}_f\rangle_f\vec{V}_f - \frac{1}{2}\rho_f^0\langle(v_f - V_\infty)^2\Delta\vec{v}_f\rangle_f$$
$$- \rho_f^0\vec{V}_\infty \cdot \langle(\vec{v}_f - \vec{V}_\infty)\Delta\vec{v}_f\rangle_f - \rho_f^0\left\langle\left(\frac{\partial\phi}{\partial t} - \frac{\partial\phi_\infty}{\partial t}\right)\Delta\vec{v}_f\right\rangle_f, \quad [A.1]$$

where

$$\vec{V}_f = \langle\vec{v}_f\rangle_f,$$

is the mean velocity and

$$\Delta\vec{v}_f = \vec{v}_f - \vec{V}_f$$

is the fluctuating velocity field. Moreover, any flow $\vec{v}_f$ verifies:

$$\langle v_f^2 \vec{v}_f \rangle_f = (V_f^2 + \langle \Delta v_f^2 \rangle_f) \vec{V}_f + 2\vec{V}_f \cdot \langle \Delta\vec{v}_f \Delta\vec{v}_f \rangle_f + \langle \Delta v_f^2 \Delta\vec{v}_f \rangle_f. \qquad [A.2]$$

Now it happens that the flow of a spherical particle in a ideal fluid is quite symmetric (no wake!) and consequently

$$\vec{V}_f \equiv \vec{V}_\infty$$

One thus deduces from (A.1) and (A.2) that:

$$\left\langle \left(\tilde{p}_f + \frac{1}{2} \rho_f^0 v_f^2 \right) \vec{v}_f \right\rangle_f = \left(\langle \tilde{p}_f \rangle_f + \frac{1}{2} \rho_f^0 V_f^2 + \frac{1}{2} \rho_f^0 \langle \Delta v_f^2 \rangle_f \right) \vec{V}_f - \rho_f^0 \left\langle \left(\frac{\partial \phi}{\partial t} - \frac{\partial \phi_\infty}{\partial t} \right) \Delta\vec{v}_f \right\rangle_f.$$

It is thus remarkable that most of the fluctuations which separately contribute to $\langle v^2 \vec{v} \rangle$ and $\langle \tilde{p}\, \vec{v} \rangle$ disappear in the above pressure and velocity combination (which is in fact the one occurring in the fluid energy flux). Moreover, in the case of rigid particles, the perturbed flow is always such that

$$\frac{\partial \phi}{\partial t} - \frac{\partial \phi_\infty}{\partial t} = -\vec{V}_s \cdot \Delta\vec{v}_f.$$

As a consequence one finally gets for rigid spherical particles in a ideal fluid:

$$\langle \tilde{p}_f \rangle_f = p_\infty - \tfrac{1}{2} \rho_f^0 \langle \Delta v_f^2 \rangle_f \qquad [A.3]$$

and

$$\left\langle \left(\tilde{p}_f + \frac{1}{2} \rho_f^0 v_f^2 \right) \vec{v}_f \right\rangle_f = \left(\langle \tilde{p}_f \rangle_f + \frac{1}{2} \rho_f^0 V_f^2 + \frac{1}{2} \rho_f^0 \langle \Delta v_f^2 \rangle_f \right) \vec{V}_f + \rho_f^0 \vec{V}_s \cdot \langle \Delta\vec{v}_f \Delta\vec{v}_f \rangle_f. \qquad [A.4]$$

The result [A.4] is rather interesting since it suggests that we need nothing more than the "Reynolds stress" $\langle \Delta\vec{v}_f \Delta\vec{v}_f \rangle$ to express the pressure-velocity correlations occurring in the energy flux of a ideal fluid containing rigid particles. This result was used in the text to get [49] for the energy flux of the mixture.